

Extensions of vector-valued Baire one functions with preservation of points of continuity^{☆,☆☆}

M. Koc^{a,*}, J. Kolář^{b,c}

^aRSJ a.s., Na Florenci 2116/15, 110 00 Praha 1, Czech Republic

^bInstitute of Mathematics, Czech Academy of Sciences, Žitná 25, 115 67 Praha 1, Czech Republic

^cMathematics Institute, University of Warwick, Coventry, UK (September 2014 – October 2015)

Abstract

We prove an extension theorem (with non-tangential limits) for vector-valued Baire one functions. Moreover, at every point where the function is continuous (or bounded), the continuity (or boundedness) is preserved. More precisely: Let H be a closed subset of a metric space X and let Z be a normed vector space. Let $f: H \rightarrow Z$ be a Baire one function. We show that there is a continuous function $g: (X \setminus H) \rightarrow Z$ such that, for every $a \in \partial H$, the non-tangential limit of g at a equals $f(a)$ and, moreover, if f is continuous at $a \in H$ (respectively bounded in a neighborhood of $a \in H$) then the extension $F = f \cup g$ is continuous at a (respectively bounded in a neighborhood of a).

We also prove a result on pointwise approximation of vector-valued Baire one functions by a sequence of locally Lipschitz functions that converges “uniformly” (or, “continuously”) at points where the approximated function is continuous.

In an accompanying paper (Extensions of vector-valued functions with preservation of derivatives), the main result is applied to extensions of vector-valued functions defined on a closed subset of Euclidean or Banach space with preservation of differentiability, continuity and (pointwise) Lipschitz property.

Keywords: vector-valued Baire one functions, extensions, non-tangential limit, continuity points, pointwise approximation, uniform convergence, continuous convergence
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1. Introduction

The purpose of this paper is to prove an extension theorem for vector-valued Baire one functions. The result is directly used in the accompanying paper [KK] where we obtain new results on extending vector-valued functions that are differentiable (or continuous, Lipschitz, ...) at some points, in a way that preserves the differentiability (continuity, Lipschitz property, ...).

Recall that a function f is *Baire one* if it is the pointwise limit of a sequence of continuous functions.

If (X, ϱ) is a metric space, $a \in X$ and $r > 0$, $B(a, r) = B_X(a, r)$ denotes the open ball $\{x \in X : \varrho(a, x) < r\}$. If $X = Z$ is a normed linear space, we sometimes use also the closed ball denoted by $\bar{B}_Z(a, r)$.

Our main result is the following theorem:

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*Corresponding author

Email addresses: martin.koc@rsj.com (M. Koc), kolar@math.cas.cz (J. Kolář)

Theorem 1.1. *Let (X, ϱ) be a metric space, $H \subset X$ a closed set, Z a normed linear space and $f: H \rightarrow Z$ a Baire one function. Then there exists a continuous function $g: (X \setminus H) \rightarrow Z$ such that*

$$\lim_{\substack{x \rightarrow a \\ x \in X \setminus H}} \|g(x) - f(a)\|_Z \frac{\text{dist}(x, H)}{\varrho(x, a)} = 0 \quad (\text{NT})$$

for every $a \in \partial H$,

$$\lim_{\substack{x \rightarrow a \\ x \in X \setminus H}} g(x) = f(a) \quad (\text{C})$$

whenever $a \in \partial H$ and f is continuous at a , and

$$g \text{ is bounded on } B(a, r) \setminus H \quad (\text{B})$$

whenever $a \in \partial H$ (or even $a \in H$), $r \in (0, \infty) \cup \{\infty\}$ and f is bounded on $B(a, 12r) \cap H$.

In [ALP, Theorem 6], the first part of the previous theorem (i.e., property (NT) without properties (C) and (B)) was proved in the special cases $Z = \mathbb{R}$ and $\dim Z < \infty$ (coordinate-wise; see [ALP, p. 607]). In [KZ], this was extended to include property (C). Our main contribution is that Z can be an arbitrary normed linear space.

Properties (NT) and (C) constitute the most important part of Theorem 1.1. Other statements (continuity of g and property (B)) are added since they might be useful and do not require much labour. The continuity of g is achieved just in the last paragraph of the proof, and from there it is obvious how a higher degree of smoothness can also be achieved when X admits a linear structure and a smooth partition of unity (cf. also [KK, Lemma 2.5 (PROVISIONAL REFERENCE)]). Property (B) requires only a bit more of attention during the proof and it is actually used (together with properties (NT) and (C)) in the accompanying paper [KK].

Theorem 1.1 is proved in Section 3. Its proof depends on Proposition 2.4 which provides an approximation of a given Baire one function f by a sequence of locally Lipschitz functions that converges “uniformly” at points of continuity of f (see property UCPC in Definition 2.1).

2. Approximation of a Baire one function by a sequence of continuous functions

In order to prove property (C) from Theorem 1.1 we need the following auxiliary notion.

Definition 2.1. Let Y, Z be metric spaces and let $h_n: Y \rightarrow Z$ ($n \in \mathbb{N}$) and $f: Y \rightarrow Z$ be arbitrary functions. We say that the pair $(\{h_n\}, f)$ has the property of *uniform convergence at points of continuity* (shortly UCPC) if the following holds: For every $y_0 \in Y$ such that f is continuous at y_0 , and for every $\varepsilon > 0$, there is $k_0 = k_0(\varepsilon) \in \mathbb{N}$ and a neighborhood $U = U(\varepsilon)$ of y_0 such that

$$\|h_k(y) - f(y_0)\|_Z < \varepsilon \quad (1)$$

for every $k \geq k_0$ and $y \in U$.

If $h_n: Y \rightarrow Z$ ($n \in \mathbb{N}$) are functions with pointwise limit $f: Y \rightarrow Z$, we say that the sequence $\{h_n\}$ has the property of *uniform convergence at points of continuity* (shortly UCPC) if the pair $(\{h_n\}, f)$ has UCPC.

Remark 2.2. Property UCPC is probably known and studied in the literature. In the terminology of [Fr, § 15], $(\{h_n\}, f)$ has UCPC if and only if h_n converges continuously to f at every $y_0 \in Y$ where f is continuous.

Directly related and more general notions are studied by Kechris and Louveau [KeL]. It is easy to show that a sequence $\{h_k\}$ with pointwise limit f has UCPC if and only if, expressed in their notation ([KeL, p. 211, 212]), $\bigcup_{\varepsilon > 0} P_{\varepsilon, f}^* \supset \bigcup_{\varepsilon > 0} P'_{\varepsilon, \{h_k\}}$, where $P = Y$.

Remark 2.3. The Hausdorff’s definition of uniform convergence at a point ([H, p. 285]) and an easy classical result [H, Theorem IV, p. 285] are not directly related to the property UCPC. The difference is that Hausdorff’s definition requires an inequality similar to (1) only for $k = k_0$ instead of $k \geq k_0$. There is a loose connection as Proposition 2.4 below is a generalization of the following corollary of [H, Theorem IV, p. 285]: If f is a Baire one function and A is the set of points of continuity of f , then there is a sequence $\{f_n\}$ of continuous functions pointwise converging to f such that $\{f_n\}$ converges (Hausdorff) uniformly exactly at every point of A . Proposition 2.4 includes our version of “uniform convergence at point”, stronger than the Hausdorff’s. Furthermore, it is generalized to vector-valued functions and strengthened to sequences of locally Lipschitz functions.

The following proposition constitutes the core part of the proof of our main result. It might be of independent interest.

Proposition 2.4. *Let Y be a metric space and Z a normed linear space. If $f: Y \rightarrow Z$ is a Baire one function then f is the pointwise limit of a sequence $\{f_n\}$ of bounded locally Lipschitz functions with UCPC such that $\{f_n\}$ is uniformly bounded on $B_Y(a, r)$ whenever $a \in Y$, $r \in (0, \infty) \cup \{\infty\}$ and f is bounded on $B_Y(a, 2r)$.*

In the case $Z = \mathbb{R}$, [ALP, p. 605] notes that the pointwise approximation of f by bounded Lipschitz functions was established in [H, § 41, pp. 264–276] and [CL, Proposition 3.9 and before].

Proof of Proposition 2.4. By the definition of Baire one functions, we can choose a sequence of continuous functions $h_n: Y \rightarrow Z$ such that $h_n \rightarrow f$ pointwise.

The proposition now follows from Lemma 2.5 and Lemma 2.6 below. \square

The following two lemmata allow to replace $\{h_n\}$ in the proof of Proposition 2.4 by a sequence with the required properties. They are independent and Lemma 2.5 can be skipped by readers not interested in properties (C) and (B) of Theorem 1.1, property UCPC and the uniform boundedness of $\{f_n\}$.

Both lemmata use the fact that every metric space Y (as well as its arbitrary subspace) is paracompact (Theorem of A. H. Stone, see, e.g., [Ru, p. 603]). Note that a topological space is called *paracompact* if every open cover of this space has a locally finite open refinement.

Lemma 2.5. *Let Y be a metric space and Z a normed linear space. Given a function $f: Y \rightarrow Z$ and a sequence of continuous functions $h_n: Y \rightarrow Z$, there exist continuous functions $\tilde{h}_n: Y \rightarrow Z$ such that*

- (a) $\tilde{h}_n(y) \rightarrow f(y)$ whenever $y \in Y$ and $h_n(y) \rightarrow f(y)$;
- (b) $\tilde{h}_n(y) \rightarrow f(y)$ whenever $y \in Y$ and f is continuous at y ;
- (c) $(\{\tilde{h}_n\}, f)$ has UCPC, in particular $\{\tilde{h}_n\}$ has UCPC provided f is a pointwise limit of $\{h_n\}$.

In a very special case (Y a compact metric space, $Z = \mathbb{R}$ and $\{h_n\}$ a bounded sequence of continuous functions converging pointwise to a function f), the lemma follows from the proof of [KeL, Theorem 2.3, p. 214–215]. Moreover, in this case, functions \tilde{h}_n are obtained as convex combinations of $\{h_n\}$.

Proof. If $M \subset Y$ is a set and $\varepsilon > 0$, denote

$$\text{int}_\varepsilon M = \{y \in Y : \text{dist}(y, Y \setminus M) \geq \varepsilon\} \subset M.$$

The set $\text{int}_\varepsilon M$ is closed. Recall that $\text{int } M$ is an open set and denotes the topological interior of M .

For $k \in \mathbb{N}$, set $\mathcal{H}_k = \{B_Z(z, 2^{-k}) : z \in Z\}$ and $\tilde{\mathcal{G}}_k = \{\text{int } f^{-1}(H) : H \in \mathcal{H}_k\}$. Then $\tilde{\mathcal{G}}_k$ is an open cover of the (open) set $Y_k := \bigcup \tilde{\mathcal{G}}_k \subset Y$.

Let \mathcal{G}_k be an open locally finite refinement of $\tilde{\mathcal{G}}_k$ that covers Y_k . It is convenient to observe that $Y_{k+1} \subset Y_k$. For every $G \in \mathcal{G}_k$, there is $z_{k,G} \in Z$ such that

$$f(G) \subset B_Z(z_{k,G}, 2^{-k}). \quad (2)$$

For $y \in Y$, the set

$$\mathcal{G}_k(y) := \{G \in \mathcal{G}_k : y \in G\}$$

is finite; if $y \in Y \setminus Y_k$ then $\mathcal{G}_k(y)$ is empty. Let

$$\mathcal{G}_k^{(j)}(y) = \{G \in \mathcal{G}_k : y \in \text{int}_{1/j} G\} \subset \mathcal{G}_k(y).$$

Note that $Y_k = \bigcup_j Y_k^{(j)}$, where $Y_k^{(j)} = \bigcup \{\text{int}_{1/j} G : G \in \mathcal{G}_k\}$. Let

$$\begin{aligned} \tilde{\Phi}_k(y) &= \bigcap \{\bar{B}_Z(z_{i,G}, 2^{-i}) : i \in \mathbb{N} \cap [1, k], G \in \mathcal{G}_i(y)\} & k \in \mathbb{N}, y \in Y, \\ \tilde{\Phi}_k^{(j)}(y) &= \bigcap \{\bar{B}_Z(z_{i,G}, 2^{-i}) : i \in \mathbb{N} \cap [1, k], G \in \mathcal{G}_i^{(j)}(y)\} & k, j \in \mathbb{N}, y \in Y, \end{aligned}$$

where we understand $\cap \emptyset = Z$. Note that $\widetilde{\Phi}_k(y) \subset \widetilde{\Phi}_k^{(j)}(y)$. Also note that

$$\widetilde{\Phi}_{k+1}^{(j)}(y) \subset \widetilde{\Phi}_k^{(j)}(y) \quad \text{and} \quad \widetilde{\Phi}_k^{(j+1)}(y) \subset \widetilde{\Phi}_k^{(j)}(y). \quad (3)$$

Later, we prove that $\widetilde{\Phi}_k^{(j)}(y)$ is a lower semi-continuous multivalued map, but we do not make such a claim for $\widetilde{\Phi}_k(y)$. Let

$$C_k := \{y \in Y : h_k(y) \in \widetilde{\Phi}_k(y)\}$$

Then C_k is a closed set. Indeed if $y_0 \notin C_k$, then the closed set $F := \widetilde{\Phi}_k(y_0)$ does not contain $h_k(y_0)$, and the same is true for $h_k(y)$ with y in a neighborhood U_1 of y_0 . Since the elements of the finite set $\mathcal{G}_i(y_0)$ ($i = 1, \dots, k$) are open, we have $\widetilde{\Phi}_k(y) \subset F$ for y in a neighborhood U_2 of y_0 . Therefore, $y_0 \in U_1 \cap U_2 \subset Y \setminus C_k$. Since $y_0 \notin C_k$ was arbitrary, C_k is a closed set.

Let

$$\Phi_k^{(j)}(y) = \begin{cases} \{h_k(y)\} & \text{if } y \in C_k \\ \widetilde{\Phi}_k^{(j)}(y) & \text{otherwise.} \end{cases} \quad (4)$$

$$(5)$$

Then $\Phi_k^{(j)}(y)$ is a closed convex set in Z , $f(y) \in \widetilde{\Phi}_k(y) \subset \widetilde{\Phi}_k^{(j)}(y)$ and $\Phi_k^{(j)}(y)$ contains $f(y)$ or $h_k(y)$. We have

$$\Phi_k^{(j)}(y) \subset \widetilde{\Phi}_k^{(j)}(y) \quad (6)$$

for every $y \in Y$ independently of which of (4), (5) happens to be true.

From the definitions and from (2),

$$\widetilde{\Phi}_k^{(j)}(y) \subset \overline{B}_Z(z_{k,G}, 2^{-k}) \subset \overline{B}_Z(f(y_0), 2^{1-k}) \quad (7)$$

whenever $G \in \mathcal{G}_k$ and $y, y_0 \in \text{int}_{1/j} G$.

By the definition of $\widetilde{\Phi}_k^{(j)}$ and since $Y \setminus \text{int}_{1/j} G$ is open for every $G \in \mathcal{G}_i$ and \mathcal{G}_i is locally finite ($i = 1, \dots, k$), we can show that the multivalued map $\widetilde{\Phi}_k^{(j)} : Y \rightarrow Z$ is lower semi-continuous. Even more, we show that for any set $M \subset Z$,

$$\left(\widetilde{\Phi}_k^{(j)}\right)^{-1}(M) = \{y \in Y : M \cap \widetilde{\Phi}_k^{(j)}(y) \neq \emptyset\} \quad \text{is open.} \quad (8)$$

This is obviously true if the set in (8) is open for every singleton $M = \{z\} \subset Z$. Let $M = \{z\} \subset Z$. If $y_0 \in Y$ is fixed so that $z \in \widetilde{\Phi}_k^{(j)}(y_0)$, let U be a neighborhood of y_0 such that $\mathcal{G}_{i,U} := \{G \in \mathcal{G}_i : G \cap U \neq \emptyset\}$ is finite for all $i = 1, \dots, k$, and let

$$H := \bigcap_{i=1}^k \bigcap_{G \in \mathcal{G}_{i,U} : z \notin \overline{B}_Z(z_{i,G}, 2^{-i})} Y \setminus \text{int}_{1/j} G.$$

Then $z \in \widetilde{\Phi}_k^{(j)}(y)$ for every $y \in U \cap H$, which is an open neighborhood of y_0 . Thus the set in (8) is indeed open.

Also $\Phi_k^{(j)}$ is lower semi-continuous. Indeed, if $O \subset Z$ is open, $y_0 \in Y$ and $\Phi_k^{(j)}(y_0) \cap O$ contains a point z , we have two cases to consider:

1. If $y_0 \in C_k$, i.e. $\Phi_k^{(j)}(y_0) = \{h_k(y_0)\}$, we note that $\widetilde{\Phi}_k^{(j)}(y_0) \supset \widetilde{\Phi}_k(y_0)$ contains $h_k(y_0) = z \in O$. Since the set in (8) is open with $M = \{z\}$ we have, for y in a neighborhood V_1 of y_0 , $z \in \widetilde{\Phi}_k^{(j)}(y) \cap O$. Since h_k is continuous, we have $\{h_k(y)\} \subset O$ for y in a neighborhood V_2 of y_0 . Thus, for $y \in V_1 \cap V_2$, $\Phi_k^{(j)}(y) \cap O \neq \emptyset$, independently of which of (4), (5) applies at y .

2. If $y_0 \in Y \setminus C_k$, then we use that (5) applies on the open set $Y \setminus C_k$ and that the set in (8) is open with $M = O$.

Hence, $\Phi_k^{(j)}$ is lower semi-continuous and from [Mi, Lemma 4.1]¹ it follows that there exists a continuous function $\varphi_k^{(j)} : Y \rightarrow Z$ such that

$$\varphi_k^{(j)}(y) \in \Phi_k^{(j)}(y) + B_Z(0, 2^{-k}) \quad (9)$$

¹p. 368. \mathcal{K} denotes (say, nonempty) convex sets (see p. 367).

for every $y \in Y$.

Obviously,

$$\left\| \varphi_k^{(j)}(y) - h_k(y) \right\|_Z < 2^{-k} \quad (10)$$

for every $y \in C_k$.

Let $\tilde{h}_k = \varphi_k^{(k)}$.

Now we want to show that for every $y_0 \in \bigcap_i Y_i$ and $i_0 \in \mathbb{N}$, there is k_1 and a neighborhood U of y_0 such that

$$\left\| \tilde{h}_k(y) - f(y_0) \right\|_Z \leq 2^{2-i_0} \quad (11)$$

whenever $k \geq k_1$ and $y \in U$.

Let $y_0 \in \bigcap_i Y_i$. Since $Y_{i_0} = \bigcup_j Y_{i_0}^{(j)}$, there is $j_0 \in \mathbb{N}$ such that $y_0 \in Y_{i_0}^{(j_0)}$. Thus, there is $G \in \mathcal{G}_{i_0}$ such that $y_0 \in \text{int}_{1/j_0} G$. Obviously, $U := B_Y(y_0, 1/2j_0) \subset \text{int}_{1/2j_0} G$. If $k \geq k_1 := \max(i_0, 2j_0)$, we have by (9) that, for every $y \in U$, the set $\tilde{h}_k(y) + B_Z(0, 2^{-k})$ intersects

$$\Phi_k^{(k)}(y) \stackrel{(6)}{\subset} \tilde{\Phi}_k^{(k)}(y) \stackrel{(3)}{\subset} \tilde{\Phi}_{i_0}^{(2j_0)}(y) \stackrel{(7)}{\subset} \bar{B}_Z(f(y_0), 2^{1-i_0}),$$

and hence

$$\left\| \tilde{h}_k(y) - f(y_0) \right\|_Z \leq 2^{1-i_0} + 2^{-k} \leq 2^{2-i_0}.$$

This finishes the proof of (11).

Assume $y_0 \in Y$ and $h_n(y_0) \rightarrow f(y_0)$. We want to show that $\tilde{h}_n(y_0) \rightarrow f(y_0)$.

1. If $y_0 \notin \bigcap_i Y_i$, then there is k_0 such that $y_0 \notin Y_k$ for all $k \geq k_0$. We have

$$\tilde{\Phi}_k(y_0) = \tilde{\Phi}_{k_0}(y_0) =: B \quad \text{for all } k \geq k_0. \quad (12)$$

The set B (defined by (12)) is an intersection of a finite number of (closed) balls with $f(y_0)$ in the interior — recall the inclusion $f(G) \subset B_Z(z_{k,G}, 2^{-k})$ (see (2)). Since $h_k(y_0) \rightarrow f(y_0)$, there is $k_1 \geq k_0$ such that, for all $k \geq k_1$, $h_k(y_0) \in B$, $y_0 \in C_k$ (see (12)) and $\left\| \tilde{h}_k(y_0) - h_k(y_0) \right\|_Z < 2^{-k}$ by (10). Since $h_k(y_0) \rightarrow f(y_0)$, we have also $\tilde{h}_k(y_0) \rightarrow f(y_0)$.

2. Assume that $y_0 \in \bigcap_i Y_i$ and $i_0 \in \mathbb{N}$ is given. Then we have (11) for $y = y_0$ and for k large enough. Since i_0 was arbitrary, we proved $\tilde{h}_k(y_0) \rightarrow f(y_0)$.

Assume now that f is continuous at $y_0 \in Y$. Then $y_0 \in \bigcap_i Y_i$ by the definition of $\tilde{\mathcal{G}}_k$, \mathcal{G}_k and Y_k . Therefore, $\tilde{h}_k(y_0) \rightarrow f(y_0)$ by the previous paragraph. Moreover, for every i_0 , there is k_1 and a neighborhood U of y_0 such that (11) holds true for all $k \geq k_1$ and $y \in U$. Since y_0 was an arbitrary point of continuity of f , this proves that $(\{\tilde{h}_k\}, f)$ has property UCPC. \square

Lemma 2.6. *Let Y be a metric space and Z a normed linear space. Given a function $f: Y \rightarrow Z$ and a sequence of continuous functions $h_n: Y \rightarrow Z$, there exist bounded locally Lipschitz functions $f_n: Y \rightarrow Z$ such that*

- (a) $f_n(y) \rightarrow z$ whenever $y \in Y$ and $h_n(y) \rightarrow z \in Z$;
- (b) $(\{f_n\}, f)$ has UCPC if $(\{h_n\}, f)$ has UCPC;
- (c) $\{f_n\}$ is uniformly bounded on $B_Y(a, r)$ whenever $a \in Y$, $r \in (0, \infty) \cup \{\infty\}$, $h_n \rightarrow f$ pointwise on $B_Y(a, 2r)$ and f is bounded on $B_Y(a, 2r)$.

Proof. The boundedness is very easy to achieve. Let $\tilde{h}_n = P_n \circ h_n$ where P_n is the radial projection of Z onto $\bar{B}_Z(0, n)$:

$$P_n(z) = \begin{cases} z, & z \in B_Z(0, n); \\ nz / \|z\|_Z, & z \in Z \setminus B_Z(0, n). \end{cases} \quad (13)$$

Every \tilde{h}_n is obviously bounded while it retains other properties mentioned in Lemma 2.6(a) and (b). Henceforth we label them h_n and assume they are bounded.

Now we provide the local uniform boundedness property requested by Lemma 2.6(c) which is slightly more complicated. (Note that this is needed only in supplementary parts of our application to differentiable extensions [KK].)

For $n \in \mathbb{N}$, let $O_n = \text{int}\{x \in Y : h_n(x) \rightarrow f(x) \text{ and } f(x) \in B_Z(0, n)\}$,

$$\phi_n(x) = \begin{cases} (n+1) + 1/\text{dist}(x, Y \setminus O_n) & \text{if } x \in O_n \\ \infty & \text{if } x \in Y \setminus O_n, \end{cases}$$

and

$$r(x) = \inf\{\phi_n(x) : n \in \mathbb{N}\} \geq 1, \quad x \in Y.$$

Note that for every $x \in Y$ there is n_0 such that $\|h_n(x)\|_Z \leq r(x)$ for every $n \geq n_0$. Indeed, if $h_n(x) \rightarrow f(x)$ and m_0 is the least integer such that $f(x) \in B_Z(0, m_0)$ then $x \in O_n$ for no $n < m_0$, hence (regardless if $x \in O_n$ or not, for various $n \geq m_0$)

$$r(x) = \inf_{n \geq 1} \phi_n(x) = \inf_{n \geq m_0} \phi_n(x) \geq \inf_{n \geq m_0} n + 1 = m_0 + 1 > \|f(x)\|_Z \leftarrow \|h_n(x)\|_Z.$$

Otherwise, $r(x) = \infty$ by the definition of O_n and ϕ_n .

If $r(x) < \infty$, then r is bounded on a neighborhood U of x (because each ϕ_n is continuous) and (since $\phi_n \geq n+1$) there is n_0 such that $r = \min(\phi_1, \phi_2, \dots, \phi_{n_0})$ on U . Hence r is continuous at x . If $r(x) = \infty$ then, for every $n \in \mathbb{N}$, $x \in Y \setminus O_n$ and hence $\phi_n(\cdot) \geq (n+1) + 1/\varrho(\cdot, x)$. Therefore $r(\cdot) \geq (n+1) + 1/\varrho(\cdot, x)$ and r is again continuous at x .

If P_r ($r \geq 1$) is the radial projection from (13), then the map $(z, r) \rightarrow P_r(z)$ is 1-Lipschitz on $Z \times [1, \infty)$ and continuous on $Z \times [1, \infty]$. Indeed, its continuity at every point $(z, \infty) \in Z \times [1, \infty]$ is obvious.

Letting

$$\tilde{h}_n(x) = P_{r(x)}(h_n(x)) \tag{14}$$

we obtain a new sequence of functions that retains the boundedness and continuity properties of $\{h_n\}$. We show that it also retains pointwise convergence as required by Lemma 2.6(a). Assume that $x \in Y$, $z \in Z$ and $h_n(x) \rightarrow z$. We need to check that $\tilde{h}_n(x) \rightarrow z$. If $r(x) = \infty$, it is obvious from (14). If $r(x) < \infty$ then we need to look at the values of $\phi_n(x)$. If $n \in \mathbb{N}$ and $\phi_n(x)$ is finite then necessarily $x \in O_n$, $f(x) = z$, and hence $\|z\|_Z < n < \phi_n(x)$. Thus we get $\|z\|_Z < \phi_n(x)$ for every n and $\|z\|_Z \leq r(x)$. This concludes the proof of $\tilde{h}_n(x) \rightarrow z$.

Assume now that $(\{h_n\}, f)$ has UCPC. We will prove that $(\{\tilde{h}_n\}, f)$ has UCPC. Assume that f is continuous at $y_0 \in Y$ and $\varepsilon \in (0, 1)$. Then $h_n(y_0) \rightarrow f(y_0)$ (cf. (1)) and there is $k_0 \in \mathbb{N}$ and a neighborhood U of y_0 such that (1) is true for $k \geq k_0$ and $y \in U$. We choose n_0 to be the least integer such that $f(y_0) \in B_Z(0, n_0)$. Then there is a neighborhood V of y_0 such that, for every $y \in V$, $f(y) \in B_Z(0, n_0) \setminus B_Z(0, n_0 - 2)$, $y \notin O_{n_0-2}$ and

$$r(y) \geq (n_0 - 1) + 1. \tag{15}$$

For $y \in U \cap V$ and $k \geq \max(k_0, n_0)$ we have $\tilde{h}_k(y) = h_k(y)$ and hence (1) is true also when h_k is replaced by \tilde{h}_k . Hence $(\{\tilde{h}_n\}, f)$ has UCPC if $(\{h_n\}, f)$ has UCPC.

Finally we want to prove that $\{\tilde{h}_n\}$ from (14) satisfies the local uniform boundedness property requested by Lemma 2.6(c). Assume that $a \in Y$, $r \in (0, \infty) \cup \{\infty\}$, $h_n \rightarrow f$ pointwise on $W := B_Y(a, 2r)$ and there is $p_0 \in \mathbb{N}$ such that $\|f(y)\|_Z < p_0$ for all $y \in W$. Then obviously $W \subset O_{p_0}$. For every $y \in B_Y(a, r)$, we have $\text{dist}(y, Y \setminus O_{p_0}) \geq r$ and, by (14) and (13), $\|\tilde{h}_n(y)\|_Z \leq r(y) \leq \phi_{p_0}(y) \leq M := p_0 + 1 + 1/r$.

This closes the part of the proof where we obtained $\{\tilde{h}_n\}$ with the local uniform boundedness property (Lemma 2.6(c)) while boundedness and the properties mentioned in Lemma 2.6(a) and (b) are retained by $\{\tilde{h}_n\}$.

Note that “locally Lipschitz” is the only property that we miss at this point. To replace the functions by locally Lipschitz ones is rather straightforward when the paracompactness of Y is used. Though, formal argument takes at least several lines for each of the properties.

Let $n \in \mathbb{N}$ and let \mathcal{U}_n be an open locally finite refinement of open cover

$$\{B_Y(x, \delta/2) : x \in Y, \delta \in (0, 1/n) \text{ such that } \tilde{h}_n(B_Y(x, \delta)) \subset B_Z(\tilde{h}_n(x), 1/n)\}.$$

Choose $x_{n,U} \in U$ for every $U \in \mathcal{U}_n$. Let $w_{n,Y}(y) = 1$ for all $y \in Y$ (or, suppose $Y \notin \mathcal{U}_n$). For $y \in Y$ and $U \in \mathcal{U}_n \setminus \{Y\}$ let

$$\mathcal{U}_n(y) = \{V \in \mathcal{U}_n : y \in V\},$$

$$w_{n,U}(y) = w_U(y) = \text{dist}(y, Y \setminus U),$$

$$W_n(y) = \sum_{U \in \mathcal{U}_n(y)} w_{n,U}(y), \quad (16)$$

$$f_n(y) = \sum_{U \in \mathcal{U}_n(y)} \frac{w_{n,U}(y)}{W_n(y)} \tilde{h}_n(x_{n,U}). \quad (17)$$

Then W_n is locally bounded away from zero, the sums in (16), (17) are locally finite, $w_{n,U}$, W_n and f_n are locally Lipschitz and locally bounded. For every $y \in Y$, we have

$$f_n(y) \in \text{conv } \tilde{h}_n(\{x_{n,U} : U \in \mathcal{U}_n(y)\}) \subset \text{conv } \tilde{h}_n(\bigcup \mathcal{U}_n(y)) \subset B_Z(\tilde{h}_n(y), 2/n),$$

and thus

$$\|f_n(y) - \tilde{h}_n(y)\|_Z \leq 2/n. \quad (18)$$

Therefore $f_n(y) \rightarrow z$ whenever $\tilde{h}_n(y) \rightarrow z \in Z$ (which implies $\tilde{h}_n(y) \rightarrow z$).

If $\tilde{h}_n(y)$ are uniformly bounded on any given set, so are f_n by (18). Since we already have validity of Lemma 2.6(c) for $\{\tilde{h}_n\}$, it is also valid for $\{f_n\}$ defined by (17).

It remains to prove (b). Assume $(\{h_n\}, f)$ has UCPC. Then we already know that $(\{\tilde{h}_n\}, f)$ has UCPC. Let $y_0 \in Y$, $\varepsilon > 0$, $k_0 \in \mathbb{N}$ and $\delta > 0$ be given such that

$$\|\tilde{h}_k(y) - f(y_0)\|_Z < \varepsilon \quad (19)$$

for every $k \geq k_0$ and $y \in B_Y(y_0, \delta)$. Let $n_0 \in \mathbb{N}$ satisfy $2/n_0 < \varepsilon$ and $n_0 \geq k_0$. Then, by (18) and (19),

$$\|f_k(y) - f(y_0)\|_Z < 2\varepsilon \quad (20)$$

for every $k \geq n_0$ and $y \in B_Y(y_0, \delta)$. Thus, $(\{f_k\}, f)$ has UCPC if $(\{h_k\}, f)$ has UCPC. We see that we obtained locally Lipschitz functions while boundedness and the properties mentioned in Lemma 2.6(a), (b) and (c) are retained. \square

3. Extensions of Baire one functions

Now we use Proposition 2.4 and an elaborated refinement of the method of [ALP, Theorem 6] to prove Theorem 1.1.

Proof of Theorem 1.1. Applying Proposition 2.4 to $Y := H$ and $f : H \rightarrow Z$, we get a sequence $f_n : H \rightarrow Z$ of bounded locally Lipschitz functions converging pointwise to f on H such that $(\{f_n\}, f)$ has property UCPC and such that $\{f_n\}$ is uniformly bounded on $B(a, 6r) \cap H$ whenever $a \in H$, $r \in (0, \infty) \cup \{\infty\}$ and f is bounded on $B(a, 12r) \cap H$.

Let $1 \leq M_1 \leq M_2 \leq \dots$ be such that $\sup_{y \in H} \|f_n(y)\|_Z \leq M_n$. For every $x \in X \setminus H$, we select a point $u(x) \in H$ with $\varrho(x, u(x)) < 2 \text{dist}(x, H)$. Then, for every $a \in H$ and $x \in X \setminus H$,

$$\text{dist}(x, H) \leq \varrho(x, a) \quad \text{and} \quad \varrho(a, u(x)) \leq 3\varrho(a, x). \quad (21)$$

Indeed, $\varrho(a, u(x)) \leq \varrho(a, x) + \varrho(x, u(x)) \leq \varrho(a, x) + 2 \text{dist}(x, H) \leq 3\varrho(a, x)$. For $x \in X \setminus H$, let

$$K_{x,n} = \max(1, \text{Lip } f_n|_{B_H(u(x), (nM_n+2) \text{dist}(x, H))}), \quad n \in \mathbb{N}. \quad (22)$$

Note that $K_{x,n}$ might be infinite. Define $1/\infty = 0$. Let $n(x)$ be the largest $n \in \mathbb{N}$ such that

$$\text{dist}(x, H) < (nK_{x,n}(nM_n + 2))^{-1} \quad (23)$$

and let $n(x) = 0$ if no such $n \in \mathbb{N}$ exists. Since $K_{x,n} \geq 1$, $M_n \geq 1$ and $\text{dist}(x, H) > 0$, there are only finitely many n satisfying (23). We claim that, for every $a \in \partial H$,

$$\lim_{\substack{x \rightarrow a \\ x \in X \setminus H}} n(x) = \infty. \quad (24)$$

To show that, let $a \in \partial H$ and $n_0 \in \mathbb{N}$ be fixed. Then there is $\eta > 0$ such that

$$K := \max(1, \text{Lip } f_{n_0}|_{B_H(a, (n_0 M_{n_0} + 2 + 3)\eta)})$$

is finite. If $x \in X \setminus H$ and $\varrho(x, a) < \eta$, (21) shows that

$$B_H(u(x), (n_0 M_{n_0} + 2) \text{dist}(x, H)) \subset B_H(a, (n_0 M_{n_0} + 2 + 3)\eta)$$

and $K_{x, n_0} \leq K < \infty$. If, moreover, $\varrho(x, a) < \lambda := (n_0 K (n_0 M_{n_0} + 2))^{-1}$, we see that (23) is satisfied with $n = n_0$, and hence $n(x) \geq n_0$. Therefore indeed $n(x) \geq n_0$ for all $x \in X \setminus H$ such that $\varrho(x, a) < \min(\eta, \lambda)$.

Let $f_0(y) = 0 \in Z$ for $y \in H$ and define

$$g(x) = f_{n(x)}(u(x)), \quad x \in X \setminus H.$$

If $a \in H$, $r \in (0, \infty) \cup \{\infty\}$ and f is bounded on $B(a, 12r) \cap H$, we have $\{f_n\}$ uniformly bounded on $B(a, 6r) \cap H$. Then, by (21), g is obviously bounded on $B_X(a, 2r) \setminus H$. This proves property (B) for g , even with $B(a, r)$ replaced by $B(a, 2r)$.

We prove that if $a \in H$, $x \in X \setminus H$ and $n(x) > 0$ then

$$\|g(x) - f(a)\|_Z \frac{\text{dist}(x, H)}{\varrho(x, a)} \leq \frac{1}{n(x)} + \frac{\|f(a)\|_Z}{n(x)} + \|f_{n(x)}(a) - f(a)\|_Z. \quad (25)$$

Since $f_n(a) \rightarrow f(a)$ and (24) is true for every $a \in \partial H$, this will prove (NT). Denote $n_x := n(x)$. We distinguish between two cases.

If $\frac{\text{dist}(x, H)}{\varrho(x, a)} \leq \frac{1}{n_x M_{n_x}}$ then we have $\|g(x) - f(a)\|_Z \frac{\text{dist}(x, H)}{\varrho(x, a)} \leq \|f_{n_x}(u(x)) - f(a)\|_Z \cdot \frac{1}{n_x M_{n_x}} \leq \frac{M_{n_x}}{n_x M_{n_x}} + \frac{\|f(a)\|_Z}{n_x M_{n_x}}$ and thus (25) holds true.

If $\frac{\text{dist}(x, H)}{\varrho(x, a)} > \frac{1}{n_x M_{n_x}}$ then

$$\varrho(u(x), a) \leq \varrho(u(x), x) + \varrho(x, a) < 2 \text{dist}(x, H) + n_x M_{n_x} \text{dist}(x, H) = (n_x M_{n_x} + 2) \text{dist}(x, H) < 1/(n_x K_{x, n_x}) \quad (26)$$

by (23), and hence

$$\begin{aligned} \|g(x) - f(a)\|_Z &= \|f_{n_x}(u(x)) - f(a)\|_Z \leq \|f_{n_x}(u(x)) - f_{n_x}(a)\|_Z + \\ &+ \|f_{n_x}(a) - f(a)\|_Z \stackrel{(22)}{\leq} K_{x, n_x} \varrho(u(x), a) + \|f_{n_x}(a) - f(a)\|_Z \stackrel{(26)}{<} \frac{1}{n_x} + \|f_{n_x}(a) - f(a)\|_Z. \end{aligned}$$

Since $\text{dist}(x, H) \leq \varrho(x, a)$, this implies (25). This completes the proof of (NT).

Now we want to prove (C). Suppose that f is continuous at $a \in \partial H$. Let $\varepsilon > 0$ be given. Applying the property UCPC of $\{f_k\}$ at $y_0 = a$ (see (1)) we obtain $\delta > 0$ and $k_0 \in \mathbb{N}$ such that

$$\|f_k(y) - f(a)\|_Z < \varepsilon \quad (27)$$

for every $k \geq k_0$ and $y \in B_H(a, \delta)$. By (24), there is $\delta_1 > 0$ such that $n(x) \geq n_0$ whenever $x \in X \setminus H$ and $\varrho(x, a) < \delta_1$. Now, if $x \in X \setminus H$ and $\varrho(x, a) < \min(\delta_1, \delta)/3$ then, by (21), $\varrho(u(x), a) \leq 3\varrho(x, a) < \min(\delta_1, \delta)$ and therefore, by (27),

$$\|g(x) - f(a)\|_Z = \|f_{n(x)}(u(x)) - f(a)\|_Z < \varepsilon.$$

Therefore, we have (C) whenever $a \in \partial H$ and f is continuous at a .

So far g has all required properties but being continuous. Let

$$\mathcal{U} = \{B_{X \setminus H}(x, \text{dist}(x, H)/3) : x \in X \setminus H\}.$$

Using the paracompactness of $(X \setminus H, \varrho)$, let $\{\phi_\alpha\}_{\alpha \in \mathcal{A}}$ be a continuous locally finite partition of unity subordinated to \mathcal{U} , (see e.g., [D, Theorem VIII.4.2]). For $\alpha \in \mathcal{A}$, find $x_\alpha \in X \setminus H$ such that $\text{supp } \phi_\alpha \subset B_{X \setminus H}(x_\alpha, \text{dist}(x_\alpha, H)/3)$. Let

$$\tilde{g}(x) = \sum_{\alpha \in \mathcal{A}} \phi_\alpha(x) g(x_\alpha) \quad x \in X \setminus H.$$

Then \tilde{g} is continuous on $X \setminus H$. Whenever g satisfies (C) resp. (B), the same is true for \tilde{g} (with $B(a, 2r) \setminus H$ for g replaced by $B(a, r) \setminus H$ for \tilde{g}). If $\alpha \in \mathcal{A}$, $x \in B_{X \setminus H}(x_\alpha, \text{dist}(x_\alpha, H)/3)$ and $a \in \partial H$ then

$$\frac{1}{4} \frac{\text{dist}(x, H)}{\varrho(x, a)} \leq \frac{\text{dist}(x_\alpha, H)}{\varrho(x_\alpha, a)} \leq 4 \frac{\text{dist}(x, H)}{\varrho(x, a)}.$$

Hence from (NT) for g we obtain that (NT) is also true for \tilde{g} . □

References

- [ALP] V. Aversa, M. Laczkovich, D. Preiss, *Extension of differentiable functions*, Comment Math. Univ. Carolin. 26 (1985) 597–609.
- [CL] A. Czászár, M. Laczkovich, *Some remarks on discrete Baire classes*, Acta Math. Acad. Sci. Hung. 33 (1979), 51–70.
- [D] J. Dugunji, *Topology*, Allyn and Bacon Series in Advanced Mathematics, University of Chicago, 1966.
- [Fr] O. Frink, *Topology in lattices*, Trans. Amer. Math. Soc. 51 (1942) 569–582.
- [H] F. Hausdorff, *Set theory*, Chelsea (1962).
- [KeL] A.S. Kechris, A. Louveau, *A classification of Baire class 1 functions*, Trans. Amer. Math. Soc. 318 (1990) 209–236.
- [KK] M. Koc, J. Kolář, *Extensions of vector-valued functions with preservation of derivatives*, in preparation or submitted.
- [KZ] M. Koc, L. Zajíček, *A joint generalization of Whitney's C^1 extension theorem and Aversa-Laczkovich-Preiss' extension theorem*, J. Math. Anal. Appl. 388 (2012) 1027–1039.
- [Mi] E. Michael, *Continuous Selections I*, The Annals of Mathematics, Second Series, 63:2 (1956) 361–382.
- [Ru] M. E. Rudin, *A new proof that metric spaces are paracompact*, Proc. Amer. Math. Soc. 20:2 (1969) 603.